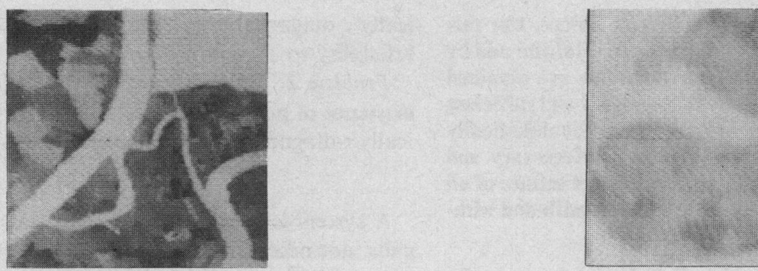
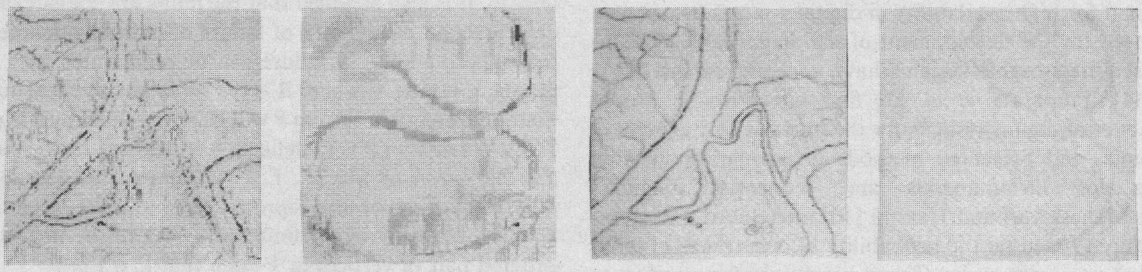


(a) (b)

Fig. 4. Results of smoothing Fig. 1(a), (b) using the maximal neighborhood of each point as an averaging neighborhood.



(a) (b) (c) (d)

Fig. 5. Edges detected on Fig. 1(a), (b) using pairs of adjacent, nonoverlapping SPAN neighborhoods (a), (b), and using the Roberts cross operator (c), (d).

As the examples show, there are real-world classes of pictures that can be treated as piecewise constant for purposes of SPAN construction. On the other hand, for a picture that contains a significant gray level ramp, the SPAN method breaks down, since it will attempt to approximate the ramp by a staircase.

It should be possible, in principle, to generalize the SPAN approach to a wider class of pictures, e.g., pictures that are approximately piecewise linear, rather than piecewise constant, in gray level. Here, for each neighborhood $N_r(x, y)$, we would test the hypothesis that its gray level population is a good fit to a ramp, say in the least squares sense. The largest r for which this fit is sufficiently good would define the neighborhood $N(x, y)$, and we could then find the maximal $N(x, y)$'s as above. On region growing in piecewise linear pictures, see [7].

In conclusion, the SPAN approach is a useful generalization of Blum's MAT concept to noisy, unsegmented pictures. Like the MAT, it provides natural, concise approximations to such pictures that can be used for purposes of encoding, recognition, and description, while avoiding the commitment of segmentation. Since it is a parallel method, it could be implemented quite efficiently on a parallel array processor.

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Some Existence Theorems for Probabilistically Diagnosable Systems

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Abstract—**This correspondence is concerned with probabilistic fault diagnosis for digital systems. The model considered in this correspondence is the diagnostic model introduced by Maheshwari**

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and Hakimi where each unit has a probability of failure. For this model under both fault assumptions by Maheshwari–Hakimi and by Barsi–Grandoni–Maestrini, some existence theorems are obtained for probabilistically diagnosable systems. 1) Necessary and sufficient conditions for the existence of testing links to form probabilistically t-diagnosable systems with and without repair. 2) Necessary and sufficient conditions for the existence of probabilities of failure of all units to form probabilistically t-diagnosable systems with and without repair which have no hardcore.

Index Terms—**Automatic diagnosis, digital systems, graphs, probabilistic fault diagnosis, probability of failure, self-diagnosable systems, testing links.**

I. Introduction

The demand for high availability in digital systems has created an urgent need for the development of self-diagnosable systems. Studies in self-diagnosable systems have appeared in the literatures [1]–[11]. Preparata *et al.* [3] first introduced a graph-theoretic model of digital systems for the purpose of diagnosis of multiple faults, and presented methods of optimal connection assignments for instantaneous and sequential diagnosis procedures. Maheshwari and Hakimi [10] introduced a diagnosability measure t based on the probability of occurrence of faults and presented necessary and sufficient conditions for a system to be probabilistically t-diagnosable in the graph-theoretic model.

For the class of failures, most of the previous works considered the following fault assumption.

Fault Assumption A: Assume that unit u_i tests unit u_j. Then, the test outcome is "0" if, under the hypothesis that u_i is fault-free, u_j is also fault-free. The test outcome is "1" if, under the same hypothesis, u_j is faulty. In the case that u_i is faulty, the test outcome is unreliable and can assume either of the values 0, 1, regardless of the status of u_j.

Barsi *et al.* [11] considered the following fault assumption.

Fault Assumption B: Assume that unit u_i tests unit u_j. Then, the test outcome is "0" if, under the hypothesis that u_i is fault-free, u_j is also fault-free. The test outcome is "1" if, under the same hypothesis, u_j is faulty. If u_i is faulty and u_j is fault-free, both test outcomes are possible. If u_i and u_j are faulty, the test outcome is necessary "1."

In this correspondence, we consider the Maheshwari–Hakimi model and four classes of probabilistically diagnosable systems as follows.

1) A class of systems which are probabilistically t-diagnosable *with* repair under fault assumption A.

2) A class of systems which are probabilistically t-diagnosable *without* repair under fault assumption A.

3) A class of systems which are probabilistically t-diagnosable *with* repair under fault assumption B.

4) A class of systems which are probabilistically t-diagnosable *without* repair under fault assumption B.

For the above classes of systems, we treat two fundamental problems in this correspondence. Suppose a system S with units $u_1, u_2, \cdots, u_n$ and the probability $p(u_i)$ of u_i being faulty for all u_i, then it seems to be an interesting problem to decide whether we can design a probabilistically t-diagnosable system by adding some testing links to S. This is Problem 1.

Problem 1: To find necessary and sufficient conditions for the existence of testing links to form probabilistically t-diagnosable systems.

Suppose a system with testing links, then it also seems to be an interesting problem to decide whether we can design a probabilistically t-diagnosable system without hardcore by giving suitable reliability to each unit. This is Problem 2.

Problem 2: To find necessary and sufficient conditions for the existence of probabilities of failure of all units to form probabilistically t-diagnosable systems without hardcore.

II. Diagnostic Model

A system is supposed to be partitioned into n subsystems, or *units*, not necessarily identical but powerful enough to test other units of the system by applying stimuli and observing the ensuing responses. Note that no unit tests itself. Let $V = \{u_1, u_2, \cdots, u_n\}$ be the set of all units. The relation of testability can be represented by a function Γ mapping V into V such that $u_j \in \Gamma(u_i)$ if and only if u_i tests u_j. It is also assumed that the faults are statistically independent and the probability of failure of unit u_i is denoted by $p(u_i)$, that is, probability of failure can be represented by a function p mapping V into R, where R is the set of positive real numbers less than one. Hence, a system S will be represented by a triple $S = (V, \Gamma, p)$. When Γ or p is undefined, it is denoted by the dash, that is, $S = (V, —, p)$ or $S = (V, \Gamma, —)$. Clearly, this diagnostic model $S = (V, \Gamma, p)$ can be also represented by a vertex-weighted digraph $G = (V, \Gamma)$ with weight function p, where the arc (u_i, u_j) is an ordered pair of vertices in V such that $u_j \in \Gamma(u_i)$.

The outcomes of the test may be represented as binary weights on the arcs of G. The weight of the arc (u_i, u_j), denoted by $s(u_i, u_j)$, is "0" if u_i finds u_j to be nonfaulty, and is "1" if u_i finds u_j to be faulty. Thus a system S together with a set of test outcomes is represented by an arc-weighted digraph, which we denote by G_s. We now define two types of consistent fault sets under fault assumptions A and B. A subset F of V, which satisfies the following two conditions, is called an *A-consistent fault set* of S with respect to G_s.

Condition 1: $s(u_i, u_j) = 1$, for all u_i and u_j such that $u_j \in \Gamma(u_i)$, $u_i \in \bar{F} = V - F$, and $u_j \in F$.

Condition 2: $s(u_i, u_j) = 0$ for all u_i, u_j such that $u_j \in \Gamma(u_i)$ and $u_i, u_j \in \bar{F}$.

A subset F of V, which satisfies the above two conditions and the following condition, is called a *B-consistent fault set* of S with respect to G_s.

Condition 3: $s(u_i, u_j) = 1$ for all u_i, u_j such that $u_j \in \Gamma(u_i)$ and $u_i, u_j \in F$.

Note that, when all arcs are labeled with "0," the empty set $\varnothing$ can be a consistent fault set. This case means that all the units are nonfaulty.

Since the faults are assumed to be statistically independent, *a priori* probability that $F \subseteq V$ is the set of faults in S is given by

$$P(F) = \prod_{u_i \in \bar{F}} (1 - p(u_i)) \cdot \prod_{u_i \in F} p(u_i).$$

Definition 1: A system $S = (V, \Gamma, p)$ is said to be probabilistically t-diagnosable without repair under fault assumption A (fault assumption B) [shortly, p-t-diagnosable without repair under A (B)] if for any arc-weighted digraph $G_s = (V, \Gamma)$ there exists at most one A-consistent (B-consistent) fault set $F \subseteq V$ such that $P(F) > t$. Let A_0^t (B_0^t) be the class of systems that are p-t-diagnosable without repair under A (B).

It is easy to see that in a p-t-diagnosable system without repair, any fault set whose *a priori* probability of occurrence greater than t can be correctly identified.

Definition 2: A system $S = (V, \Gamma, p)$ is said to be probabilistically t-diagnosable with repair under fault assumption A (fault assumption B) [shortly, p-t-diagnosable with repair under A (B)] if for any arc-weighted digraph $G_s = (V, \Gamma)$ there exist no A-consistent (B-consistent) fault sets $F_1, F_2, \cdots, F_l$ $(l \geq 2)$ such that $\bigcap_{i=1}^{l} F_i = \varnothing$ and $P(F_i) > t$ for $i = 1, 2, \cdots, l$. Let A_1^t (B_1^t) be the class of systems that are p-t-diagnosable with repair under A (B).

In a p-t-diagnosable system with repair, the intersection of all consistent fault sets, whose *a priori* probability of occurrence is greater than t, is not empty so that at least the units belonging to the intersection are all faulty. Hence, there exists a sequence of applications of tests and repairs of identified faults that allows all faults originally present to be identified.

The following theorem easily follows from the above definitions.

Theorem 1: For any $0 < t < 1$, 1) $A_0^t \subseteq A_1^t$, 2) $A_0^t \subseteq B_0^t$, 3) $A_1^t \subseteq B_1^t$, and 4) $B_0^t \subseteq B_1^t$.

Let

$$K(t) = -\log t + \sum_{u_i \in V} \log (1 - p(u_i))$$

and

$$W(u_i) = \log \frac{1 - p(u_i)}{p(u_i)}$$

for all $u_i \in V$. Then we can readily show that for any fault set $F \subseteq V$, $P(F) > t$ if and only if $W(F) < K(t)$ where

$$W(F) = \sum_{u_i \in F} W(u_i).$$

Therefore, the problem of finding the faulty units in a p-t-diagnosable system is reduced to that of determining the consistent fault set for which the sum of the weights of the vertices belonging to it is less than $K(t)$. Hence, when referring to a system S represented by $S = (V, \Gamma, p)$ of its diagnostic model, the representation $S = (V, \Gamma, W)$ will also be utilized. Also, we assume that a unit is more likely to be nonfaulty than faulty, that is, $p(u_i) < \frac{1}{2}$ for all $u_i \in V$. This implies $W(u_i) > 0$ for all $u_i \in V$.

A *hardcore* of a system $S = (V, \Gamma, W)$ is defined to be a unit u_i such that $W(u_i) \geq K(t)$, that is, $P(u_i) \leq t$. Graph $\langle Z \rangle$, called the *induced* subgraph of G on Z, consists of vertex set $Z \subseteq V$ and all arcs in G incident at pairs of vertices in Z. It is convenient to further extend the domain of Γ as follows. Let

$$\Gamma^{-1}(u_i) = \{u_j \mid u_i \in \Gamma(u_j)\}.$$

For $X \subseteq V$, let

$$\Gamma(X) = \left\{ \bigcup_{u_i \in X} \Gamma(u_i) \right\} - X$$

$$\Gamma^{-1}(X) = \left\{ \bigcup_{u_i \in X} \Gamma^{-1}(u_i) \right\} - X.$$

Using the preceding definitions and notations, we can rewrite Problems 1 and 2 as follows.

Problem 1: Given a system $S = (V, —, W)$; find necessary and sufficient conditions for the existence of a function Γ to form probabilistically t-diagnosable systems.

Problem 2: Given a system $S = (V, \Gamma, —)$; find necessary and sufficient conditions for the existence of a function W to form probabilistically t-diagnosable systems without hardcore.

III. Existence Theorems for Function Γ

In this section, we consider Problem 1 for each class of p-t-diagnosable systems. First, we shall show necessary and sufficient conditions for the existence of a function Γ to form p-t-diagnosable systems in A_0^t and A_1^t.

Lemma 1: If $S = (V, \Gamma, W)$ is a system in A_1^t, then there exists no partition $\{U_1, U_2\}$ of V such that $W(U_1) < K(t)$ and $W(U_2) < K(t)$.

Proof: Suppose for some partition $\{U_1, U_2\}$ of V, $W(U_1) < K(t)$ and $W(U_2) < K(t)$. We can easily find a weighted digraph G_s of S that has U_1 and U_2 as A-consistent fault sets. Thus, by Definition 2, S would not be p-t-diagnosable with repair under fault assumption A. Q.E.D.

Lemma 2: Given a system $S = (V, —, W)$, there exists a function Γ, such that $\hat{S} = (V, \Gamma, W)$ is in A_0^t, if there exists no partition $\{U_1, U_2\}$ of V with $W(U_1)$, $W(U_2) < K(t)$.

Proof: Consider a maximally connected graph $G = (V, \Gamma)$, i.e., $\Gamma(u_i) = V$ for all $u_i \in V$. We shall prove that the system $\hat{S} = (V, \Gamma, W)$ is in A_0^t. The proof is by contradiction. Assume the existence of a weighted digraph G_s for which there are two A-consistent fault sets F_1 and F_2 with $W(F_1)$, $W(F_2) < K(t)$. Let $X = V - (F_1 \cup F_2)$, $Y = F_1 \cap F_2$, $Z_1 = F_1 - Y$, and

$$Z_2 = F_2 - Y.$$

Without loss of generality, we have $Z_1 \neq \varnothing$.

If $X = \varnothing$, then $\{F_2, Z_1\}$ is a partition of V such that $W(F_2) < K(t)$ and $W(Z_1) < K(t)$. This contradicts the hypothesis. Therefore, we have $X \neq \varnothing$.

Since $\Gamma(u_i) = V$ for all $u_i \in V$, there exists an arc (u_i, u_j) such that $u_i \in X$ and $u_j \in Z_1$. Since $u_i, u_j \in \bar{F}_2$ and F_2 is an A-consistent fault set, we have $s(u_i, u_j) = 0$. But since F_1 is also an A-consistent fault set and $u_i \in \bar{F}_1$, $u_j \in F_1$, we have $s(u_i, u_j) = 1$. This is a contradiction. Hence, our initial assumption was wrong. Q.E.D.

Theorem 2: Given a system $S = (V, —, W)$, then there exists a function Γ, such that $\hat{S} = (V, \Gamma, W)$ is in A_0^t, if and only if there exists no partition $\{U_1, U_2\}$ of V such that $W(U_1) < K(t)$ and $W(U_2) < K(t)$.

Proof: The necessity of the theorem follows from 1) of Theorem 1 and Lemma 1. The sufficiency follows from Lemma 2. Q.E.D.

Theorem 3: Given a system $S = (V, —, W)$, then there exists a function Γ, such that $\hat{S} = (V, \Gamma, W)$ is in A_1^t, if and only if there exists no partition $\{U_1, U_2\}$ of V such that $W(U_1) < K(t)$ and $W(U_2) < K(t)$.

Proof: The necessity of the theorem follows from Lemma 1. The sufficiency follows from Lemma 2 and 1) of Theorem 1. Q.E.D.

As an example, consider a system consisting of three units such that $p(u_1) = \frac{1}{5}$, $p(u_2) = \frac{1}{4}$, $p(u_3) = \frac{1}{3}$, and $t = \frac{1}{20}$. Then we have $W(u_1) = \log 4$, $W(u_2) = \log 3$, $W(u_3) = \log 2$, and $K(t) = \log 8$. Let $U_1 = \{u_1\}$ and $U_2 = \{u_2, u_3\}$, then we have $W(U_1) = \log 4 < K(t)$ and $W(U_2) = \log 6 < K(t)$. Thus, from Theorems 2 and 3, it can be seen that there exists no function Γ such that the system is in A_0^t or A_1^t.

When the probability of failure of all units is the same, it can easily be seen that the above condition means that the number of faulty units present does not exceed $\lfloor (n-1)/2 \rfloor$ where $n = |V|$ and $\lfloor x \rfloor$ denotes the greatest integer not exceeding x. Therefore, Theorems 2 and 3 contain Theorem 1 of Preparata *et al.* [3].

Next, we show the necessary and sufficient condition for the existence of a function Γ to form a p-t-diagnosable system in B_0^t.

Theorem 4: Given a system $S = (V, —, W)$, then there exists a function Γ, such that $\hat{S} = (V, \Gamma, W)$ is in B_0^t, if and only if there exists no partition $\{U, \{u_1\}, \{u_2\}\}$ of V such that $W(U) + W(u_1) < K(t)$ and $W(U) + W(u_2) < K(t)$.

Proof—Necessity: The proof is by contradiction. Suppose that $\hat{S} = (V, \Gamma, W)$ is in B_0^t, and that for some partition $\{U, \{u_1\}, \{u_2\}\}$ of V, $W(U) + W(u_1) < K(t)$ and $W(U) + W(u_2) < K(t)$. Let G_s be an arc-weighted digraph such that $s(u_i, u_j) = 1$ for all arcs (u_i, u_j). Let $F_1 = U \cup \{u_1\}$ and $F_2 = U \cup \{u_2\}$, then F_1 and F_2 are B-consistent fault sets with respect to G_s. Thus, $\hat{S}$ is not p-t-diagnosable without repair under assumption B. This contradicts the hypothesis.

Sufficiency: Consider a maximally connected digraph $G = (V, \Gamma)$. We shall prove that the system $\hat{S} = (V, \Gamma, W)$ is in B_0^t. The proof is by contradiction. Assume the existence of a weighted digraph G_s for which there are two B-consistent fault sets F_1 and F_2 with $W(F_1)$, $W(F_2) < K(t)$. Let $X = V - (F_1 \cup F_2)$, $Y = F_1 \cap F_2$, $Z_1 = F_1 - Y$, and $Z_2 = F_2 - Y$. Without loss of generality, we have $Z_1 \neq \varnothing$.

If $X \neq \varnothing$, then there exists an arc (u_i, u_j) such that $u_i \in X$ and $u_j \in Z_1$. Since $u_i, u_j \in \bar{F}_2$ and F_2 is a B-consistent fault set, we have $s(u_i, u_j) = 0$. But since F_1 is also a B-consistent fault set and $u_j \in F_1$, we have $s(u_i, u_j) = 1$. This is a contradiction. Hence, we have $X = \varnothing$.

If $|Z_1| \geq 2$, then there exists an arc (u_i, u_j) with $u_i, u_j \in Z_1$ since G is a maximally connected digraph. Since $u_i, u_j \in \bar{F}_2$ and F_2 is a B-consistent fault set, we have $s(u_i, u_j) = 0$. But since F_1 is also a B-consistent fault set and $u_j \in F_1$, we have $s(u_i, u_j) = 1$. This is a contradiction. Hence, we have $|Z_1| \leq 1$. Similarly, we have $|Z_2| \leq 1$. Since $Z_1 \neq \varnothing$ and $|Z_1| \leq 1$, we have $|Z_1| = 1$, and can let $Z_1 = \{u_1\}$.

If $Z_2 \neq \varnothing$, then $|Z_2| = 1$ and we can let $Z_2 = \{u_2\}$. Then we have

$$W(Y) + W(u_1) = W(F_1) < K(t),$$
$$W(Y) + W(u_2) = W(F_2) < K(t),$$

and $\{Y, \{u_1\}, \{u_2\}\}$ is a partition of V since $X = \varnothing$. This contradicts the hypothesis of the theorem.

If $Z_2 = \varnothing$, then $Y = F_2$. Let $U = Y - \{u_2\}$ for some $u_2 \in Y$. Then we have

$$W(U) + W(u_1) < W(F_1) < K(t),$$
$$W(U) + W(u_2) = W(F_2) < K(t),$$

and $\{Y, \{u_1\}, \{u_2\}\}$ is a partition of V since $X = \varnothing$. This contradicts the hypothesis of the theorem. Hence, our initial assumption was wrong, which implies $\hat{S} = (V, \Gamma, W)$ is in B_0^t. Q.E.D.

When the probability of failure of all units is the same, we can also show that the condition of Theorem 4 means that the number of faulty units present does not exceed $n - 2$. This coincides with Theorem 1 of Barsi *et al.* [11], which is a special case of Theorem 4.

Lastly, we show the necessary and sufficient condition for the existence of a function Γ to form a p-t-diagnosable system in B_1^t.

Theorem 5: Given a system $S = (V, —, W)$, then there exists a function Γ, such that $\hat{S} = (V, \Gamma, W)$ is in B_1^t, if and only if there exists a subset U of V such that $|U| = |V| - 1$ and $W(U) \geq K(t)$.

Proof—Necessity: The proof is by contradiction. Assume that there exists no subset U of V such that $|U| = |V| - 1$ and $W(U) \geq K(t)$. Let $V = \{u_1, u_2, \cdots, u_n\}$, and $F_i = V - \{u_i\}$ for all $u_i \in V$. Then $|F_i| = |V| - 1$ and thus $W(F_i) < K(t)$ for all $i = 1, 2, \cdots, n$. Clearly,

$$\bigcap_{i=1}^{n} F_i = \varnothing \quad \text{and} \quad \bigcup_{i=1}^{n} F_i = V.$$

Consider an arc-weighted digraph G_s for which $s(u_i, u_j) = 1$ for all arcs (u_i, u_j). Then, F_i is a B-consistent fault set with respect to G_s for all $i = 1, 2, \cdots, n$. Hence, $\hat{S} = (V, \Gamma, W)$ is not p-t-diagnosable with repair under assumption B. This contradicts the hypothesis.

Sufficiency: Consider a maximally connected digraph $G = (V, \Gamma)$, that is, $\Gamma(u_i) = V$ for all $u_i \in V$. We shall prove that $\hat{S} = (V, \Gamma, W)$ is in B_1^t. The proof is by contradiction. Assume the existence of a weighted digraph G_s for which there exist B-consistent fault sets $F_1, F_2, \cdots, F_l$ such that

$$\bigcap_{i=1}^{l} F_i = \varnothing \quad \text{and} \quad W(F_i) < K(t), \qquad \text{for all } i = 1, 2, \cdots, l.$$

If $\bigcup_{i=1}^{l} F_i \neq V$, then there exists an arc (u_i, u_j) such that $u_i \in \bigcap_{i=1}^{l} \bar{F}_i$ and $u_j \in F_j \cap \bar{F}_k$ for some $F_j, F_k \in \{F_1, F_2, \cdots, F_l\}$. Since $u_i, u_j \in \bar{F}_k$ and F_k is a B-consistent fault set, we have $s(u_i, u_j) = 0$. But since F_j is also a B-consistent fault set and $u_j \in F_j$, we have $s(u_i, u_j) = 1$. This is a contradiction. Hence, we have

$$\bigcup_{i=1}^{l} F_i = V.$$

If $|\bar{F}_j| \geq 2$ for some F_j, then there exists an arc (u_i, u_j) with $u_i, u_j \in \bar{F}_j$ since $\Gamma(u_i) = V$. $\bigcup_{i=1}^{l} F_i = V$ implies $u_j \in F_k$ for some $F_k \neq F_j$. Since $u_i, u_j \in \bar{F}_j$ and F_j is a B-consistent fault set, we have $s(u_i, u_j) = 0$. But since F_k is also a B-consistent fault set and $u_j \in F_k$, we have $s(u_i, u_j) = 1$. This is a contradiction. Hence, we have $|\bar{F}_i| \leq 1$, i.e., $|F_i| \geq |V| - 1$ for all $i = 1, 2, \cdots, l$. From this and $\bigcap_{i=1}^{l} F_i = \varnothing$, we have $F_i \neq V$ for all F_i. Thus $|F_i| = |V| - 1$, i.e., $|\bar{F}_i| = 1$ for all F_i. $\bigcap_{i=1}^{l} F_i = \varnothing$ implies $\bigcup_{i=1}^{l} \bar{F}_i = V$. Hence, $\bigcup_{i=1}^{l} \bar{F}_i = V$ and $|\bar{F}_i| = 1$ imply that $l = |V|$, i.e., all the subsets U of V with $|U| = |V| - 1$ are $F_1, F_2, \cdots, F_l$. Moreover, $W(F_i) < K(t)$. This contradicts the hypothesis that for some $U \subseteq V$ with $|U| = |V| - 1$, $W(U) \geq K(t)$. Hence, our initial assumption was wrong, which implies $\hat{S} = (V, \Gamma, W)$ is in B_1^t. Q.E.D.

Theorems 2 and 3 have shown that the existence theorem for A_0^t and A_1^t is the same. The same result cannot be obtained for B_0^t and B_1^t. That is, the condition of Theorem 4 is not equivalent to that of Theorem 5. However, when the probability of fault of all units is the same, the condition of Theorem 4 coincides with that of Theorem 5. That is, the condition for B_0^t and B_1^t can be restated that the number of faulty units present does not exceed $n - 2$ when all the faults are equiprobable. This is an interesting result.

To illustrate Theorems 4 and 5, consider a system $S = (V, —, W)$, which consists of five units $u_1, u_2, \cdots, u_5$. Suppose $p(u_1) = \frac{1}{7}$, $p(u_2) = \frac{1}{6}$, $p(u_3) = \frac{1}{5}$, $p(u_4) = \frac{1}{4}$, and $p(u_5) = \frac{1}{3}$, then we have $W(u_1) = \log 6$, $W(u_2) = \log 5$, $W(u_3) = \log 4$, $W(u_4) = \log 3$, and $W(u_5) = \log 2$.

Suppose that $t = \frac{1}{350}$, then we have $K(t) = \log 100$.

$$\min \{W(U) \mid U \subseteq V, |U| = |V| - 1\}$$
$$= W(u_2) + W(u_3) + W(u_4) + W(u_5) = \log 120 \geq K(t).$$

This implies that for $t = \frac{1}{350}$, $S = (V, —, W)$ satisfies the condition of Theorems 4 and 5. Thus there exists a function Γ such that $\hat{S} = (V, \Gamma, W)$ is in B_0^t and B_1^t.

Suppose that $t = \frac{1}{700}$, then we have $K(t) = \log 200$. Let $U = \{u_3, u_4, u_5\}$, then $W(U) + W(u_1) = \log 144 < K(t)$ and $W(U) + W(u_2) = \log 120 < K(t)$. This implies that for $t = \frac{1}{700}$, $S = (V, —, W)$ does not satisfy the condition of Theorem 4. Thus there exists no function Γ such that $\hat{S} = (V, \Gamma, W)$ is in B_0^t. However, we can see that $S = (V, —, W)$ satisfies the condition of Theorem 5 as follows: Let $U = \{u_1, u_2, u_3, u_4\}$, then we have $W(U) = \log 360 \geq K(t)$. Hence, for $t = \frac{1}{700}$, there exists a function Γ such that $\hat{S} = (V, \Gamma, W)$ is in B_1^t.

IV. Existence Theorems for Function W

In the last section, we have considered Problem 1 for each class of p-t-diagnosable systems, and have obtained different results for A_0^t, A_1^t, B_0^t, and B_1^t. However, for Problem 2, we can show that the results for A_0^t, A_1^t, B_0^t, and B_1^t are all the same.

First we need the following lemma.

Lemma 3: If $S = (V, \Gamma, W)$ is a system in B_1^t and has no hardcore, then 1) $\Gamma^{-1}(u_i) \neq \varnothing$ for all $u_i \in V$, and 2) for any $U \subseteq V$ with $|U| = 2$, we have $\Gamma^{-1}(U) \neq \varnothing$.

Proof: To see that 1) is necessary, suppose otherwise. Then there exists a vertex $u_k \in V$ such that $\Gamma^{-1}(u_k) = \varnothing$. Let G_s be an arc-weighted digraph such that $s(u_i, u_j) = 0$ for all arcs (u_i, u_j) and let $F_1 = \{u_k\}$ and $F_2 = \varnothing$. Then it can easily be seen that both F_1 and F_2 are B-consistent fault sets with respect to G_s. Moreover, since S has no hardcore, $W(F_1)$ and $W(F_2) < K(t)$. This implies that S is not p-t-diagnosable with repair under B. This contradicts the hypothesis of the theorem.

To see that 2) is necessary, suppose otherwise. Then there exists a subset U of V such that $|U| = 2$ and $\Gamma^{-1}(U) = \varnothing$. Let $U = \{u_1, u_2\}$, $F_1 = \{u_1\}$ and $F_2 = \{u_2\}$. Consider an arc-weighted digraph G_s such that for all $u_j \in \Gamma(u_i)$, $s(u_i, u_j) = 1$ if $u_j = u_1$ or u_2, and $s(u_i, u_j) = 0$ otherwise. Then, clearly F_1 and F_2 are B-consistent fault sets with respect to G_s. Moreover, $W(F_1)$ and $W(F_2) < K(t)$ since S has no hardcore. This implies that S is not in B_1^t, which contradicts the hypothesis of the theorem. Q.E.D.

Suppose that all the faults are equiprobable, that is, $p(u_i) = p$ for all $u_i \in V$. One can show that

$$\max\{p(1-P)^{n-1} \mid 0 < p < \tfrac{1}{2}, n = |V| \geq 3\} = \frac{(n-1)^{n-1}}{n^n}.$$

Hence, if

$$t \geq \frac{(n-1)^{n-1}}{n^n},$$

then $P(u_i) = p(1-p)^{n-1} \leq t$ for any $0 < p < \frac{1}{2}$, that is, each unit u_i is a hardcore for any probability of failure. As a matter of fact, this case is not realistic. So, from now on, we assume that

$$0 < t < \frac{(n-1)^{n-1}}{n^n}.$$

For any

$$0 < t < \frac{(n-1)^{n-1}}{n^n},$$

it can easily be seen that there exists a solution p to the following inequalities:

$$p^2(1-p)^{n-2} \leq t < p(1-p)^{n-1} \quad \text{and} \quad 0 < p < \tfrac{1}{2}.$$

This implies that

$$\log\frac{1-p}{p} < \log\frac{(1-p)^n}{t} \leq 2\log\frac{1-p}{p}.$$

If we let $W = \log[(1-p)/p]$, then the above inequalities imply

$$W < K(t) \leq 2W.$$

Therefore, for any

$$0 < t < \frac{(n-1)^n}{n^n},$$

there exists a function W such that $W(u_i) < K(t) \leq 2W(u_i)$ for all $u_i \in V$.

Lemma 4: Given a system $S = (V, \Gamma, —)$ and

$$0 < t < \frac{(n-1)^{n-1}}{n^n},$$

then there exists a function W, such that $\hat{S} = (V, \Gamma, W)$ has no hardcore and is in A_0^t, if 1) $\Gamma^{-1}(u_i) \neq \varnothing$ for all $u_i \in V$ and 2) $\Gamma^{-1}(U) \neq \varnothing$ for any $U \subseteq V$ with $|U| = 2$.

Proof: Since

$$0 < t < \frac{(n-1)^{n-1}}{n^n},$$

there exists a function W such that $W(u_i) < K(t) \leq 2W(u_i)$ for all $u_i \in V$. This implies that $W(u_i) + W(u_j) \geq K(t)$ for all $u_i, u_j \in V$.

Let $\hat{S} = (V, \Gamma, W)$ be a system satisfying the conditions of the theorem. We shall show that $\hat{S}$ is in A_0^t. Assume that $\hat{S}$ is not in A_0^t, then there exists an arc-weighted digraph G_s for which there are two A-consistent fault sets F_1 and F_2 with $W(F_1)$ and $W(F_2) < K(t)$. Since $W(u_i) + W(u_j) \geq K(t)$ for all $u_i, u_j \in V$, we have $|F_1| \leq 1$ and $|F_2| \leq 1$.

If $F_1 = \varnothing$, then $F_2 \neq \varnothing$ and we let $F_2 = \{u\}$. From 1) we have $\Gamma^{-1}(u) \neq \varnothing$, thus there exists a vertex $v \in \Gamma^{-1}(u)$. Since $u \in F_2$ and F_2 is an A-consistent fault set, we have $s(v, u) = 1$. But since $F_1 = \varnothing$ is also an A-consistent fault set and $u, v \in \bar{F}_1$, we have $s(v, u) = 0$. This is a contradiction. Therefore we have $F_1 \neq \varnothing$, and similarly $F_2 \neq \varnothing$.

Let $F_1 = \{u_1\}$, $F_2 = \{u_2\}$ and $U = \{u_1, u_2\}$. From 2) we have $\Gamma^{-1}(U) \neq \varnothing$, thus without loss of generality we have $u_3 \in \bar{U}$ and $u_1 \in \Gamma(u_3)$. Since $u_1, u_3 \in \bar{F}_2$ and F_2 is an A-consistent fault set, we have $s(u_3, u_1) = 0$. But since F_1 is also an A-consistent fault set and $u_3 \in \bar{F}_1$, $u_1 \in F_1$, we have $s(u_3, u_1) = 1$. This is a contradiction. Hence, our initial assumption was wrong, which implies that $\hat{S}$ is in A_0^t. Q.E.D.

Condition 4: 1) $\Gamma^{-1}(u_i) \neq \varnothing$ for all $u_i \in V$ and 2) $\Gamma^{-1}(U) \neq \varnothing$ for any $U \subseteq V$ with $|U| = 2$.

Given a system $S = (V, \Gamma, —)$ and

$$0 < t < \frac{(n-1)^{n-1}}{n^n},$$

then we have the following theorem for A_0^t, A_1^t, B_0^t, and B_1^t from Theorem 1 and Lemmas 3 and 4.

Theorem 6: There exists a function W, such that $\hat{S} = (V, \Gamma, W)$ is in A_0^t (A_1^t, B_0^t, and B_1^t) and has no hardcore if and only if S satisfies Condition 4.

Proof: In case of A_0^t, the necessity follows from 2) and 4) of Theorem 1 and Lemma 3. The sufficiency follows from Lemma 4.

In case of A_1^t, the necessity follows from 3) of Theorem 1 and Lemma 3. The sufficiency follows from 1) of Theorem 1 and Lemma 4.

In case of B_0^t, the necessity follows from 4) of Theorem 1 and Lemma 3. The sufficiency follows from 2) of Theorem 1 and Lemma 4.

In case of B_1^t, the necessity follows from Lemma 3. The sufficiency follows from 1) and 3) of Theorem 1 and Lemma 4.

Q.E.D.

Hence, by Theorem 6 we can see that the necessary and sufficient conditions for the existence of a function W to form a p-t-diagnosable system in A_0^t, A_1^t, B_0^t, and B_1^t are all the same.

V. Conclusions

This correspondence has considered two fundamental problems for four classes of p-t-diagnosable systems A_0^t, A_1^t, B_0^t, and B_1^t: 1) Given a system $S = (V, —, W)$, find necessary and sufficient conditions for the existence of a function Γ such that $\hat{S} = (V, \Gamma, W)$ is p-t-diagnosable. 2) Given a system $S = (V, \Gamma, —)$, find necessary and sufficient conditions for the existence of a function W such that $\hat{S} = (V, \Gamma, W)$ is p-t-diagnosable and has no hardcore.

The model and its diagnosability treated here are more general than those in previous investigations [1]–[11]. When the probability of failure of all units is the same, the existence theorem of a function Γ for A_0^t and A_1^t coincides with that of Preparata *et al.* [3] and that for B_0^t coincides with that of Barsi *et al.* [11]. Hence, some existence theorems in this paper are the generalizations of previous results [3], [11].

The necessary and sufficient conditions for the existence of functions Γ and W are, in general, very simple and not so hard to test. Hence, these results are useful to design various p-t-diagnosable systems. The synthesis problems of finding the optimal functions Γ and W have not been considered in this correspondence. A function Γ is said to be optimal if

$$\sum_{u_i \in V} |\Gamma(u_i)|$$

is minimized. A function W is said to be optimal if

$$\sum_{u_i \in V} W(u_i)$$

is minimized. In the above sense of optimality, optimal designs of p-t-diagnosable systems are open research problems.

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