On Closedness and Test Complexity of Logic Circuits

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***Abstract*—The concept of closedness of a set of logic functions under stuck-type faults is introduced. All sets of logic functions closed under stuck-type faults are classified. For the sets of logic functions closed under stuck-type faults, the test complexity and the universal test sets are considered. It is shown that for each class of linear functions, OR functions, and AND functions, both the minimum numbers of multiple fault detection tests and multiple fault location tests are exactly $n + 1$, where n is the number of inputs of the circuits, and that there exists universal test sets with $n + 1$ tests to detect and locate all multiple faults in such circuits.**

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***Index Terms*—Closed classes, fault diagnosis, logic functions, stuck faults, test complexity, universal test sets.**

I. Introduction

THE problem of fault detection and location in logic circuits is of extreme importance and much attention has been devoted to it. Many studies have been reported for special classes of logic functions such as fan-out-free functions [2], linear functions [2]–[4], monotone functions, and unate functions [5]–[8]. In this paper we introduce a new concept of the closedness of a set of logic functions under stuck-type faults. A set F of logic functions is said to be *closed* under stuck-type faults if any stuck fault changes a function in F to a faulty function in F, where the circuit is composed only of the elements realizing functions in F. It is shown that any set

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of logic functions closed under stuck-type faults is also closed under the degeneration of input variables and vice versa.

The classical closedness problem in logic and switching theory was completely solved for the binary case and all the closed classes under composition were classified [1]. Using the results, all the sets of logic functions closed under both degeneration and composition are classified, among which are the sets of monotone functions, linear functions, OR functions, and AND functions, etc. Although both the sets of fan-out-free functions and unate functions are not closed under composition, it is shown that these are closed under some restricted composition, and thus these sets are shown to be closed under stuck-type faults provided that the circuits are constructed by the restricted composition.

For the sets of logic functions closed under stuck faults we consider the test complexity, i.e., the minimum number of fault detection tests and fault location tests, and present the universal test sets derived only from the functional description of the circuits. The universal test set for a class F can detect and/or locate all stuck faults in any realization of a function in F such that the circuit is composed only of the elements realizing functions in F. It is shown that for each class of linear functions, OR functions, and AND functions, both the minimum numbers of multiple fault detection tests and multiple fault location tests are exactly $n + 1$, where n is the number of inputs of the circuits, and that there exist universal test sets with $n + 1$ tests to detect and locate all multiple faults in such circuits

II. Closedness Under Degeneration

In this section we prepare several notations and definitions on the closedness of logic functions and present all the sets of logic functions closed under both degeneration and composition.

Let Ω_n be the set of all n-input logic functions, then the set of all logic functions can be represented as follows:

$$\Omega = \bigcup_{n=1}^{\infty} \Omega_n.$$

The set of all single-input logic functions is

$$\Omega_1 = \{0, 1, I, N\}$$

where 0 and 1 denote the constant functions, I denotes the identity function, and N denotes the negation function, i.e., $0(x) = 0$, $1(x) = 1$, $I(x) = x$, and $N(x) = \bar{x}$.

Let f be in Ω_p and G be a set of logic functions. A logic function h in Ω_n is said to be *constructable from f and G* if h can be written as follows:

$$h(x_1, x_2, \cdots, x_n) = f(g_{i_1}(x_{t(1,1)}, \cdots, x_{t(1,n_1)}), \cdots, g_{i_p}(x_{t(p,1)}, \cdots, x_{t(p,n_p)}))$$

where the $x_{t(i,j)}$'s are variables chosen from $x_1, x_2, \cdots, x_n$ and each $g_{i_1}, \cdots, g_{i_p} \in G \cup \{I\}$. It is obvious that this construction implies fan-out-free structure except at the primary inputs. However, applying this construction many times we can obtain arbitrary network structure as follows. Let F and G be subsets of Ω. We denote $F \otimes G$ the set of all logic functions constructable from elements in F and subsets of G.

For any subset F of Ω, let

1) $F^{(1)} = F \otimes \{I\}$,
2) $F^{(d)} = F^{(d-1)} \otimes F^{(1)}$ for $d \geq 2$,
3) $F^* = \bigcup_{d=1}^{\infty} F^{(d)}$.

F^* is said to be the *closure* of F. F is said to be *closed under composition* if $F = F^*$. F is said to be *closed under degeneration* if $F = F \otimes \{0, 1\}$. F is said to be *closed under 0-degeneration* (*1-degeneration*) if $F = F \otimes \{0\}$ ($F = F \otimes \{1\}$).

Now we present the necessary and sufficient condition for a set of logic functions to be closed under both degeneration and composition in the following lemma. Henceforth, we assume that any set of logic functions except $\{0, 1\}$, $\{0\}$, and $\{1\}$ contains the identity function I, in order to consider nontrivial sets of logic functions.

Lemma 1: For any subset F of Ω such that $F = F^*$:

1) $F = F \otimes \{0, 1\}$ if and only if $0, 1 \in F$,
2) $F = F \otimes \{0\}$ if and only if $0 \in F$,
3) $F = F \otimes \{1\}$ if and only if $1 \in F$.

Proof: 1) Suppose that $F = F \otimes \{0, 1\}$, then we have

$$F = F \otimes \{0, 1\} \supseteq \{I\} \otimes \{0, 1\} \supseteq \{0, 1\}$$

which implies $0, 1 \in F$.

Conversely, suppose that $0, 1 \in F$. Clearly, we have $F = F \otimes F$. Therefore, $F = F \otimes F \supseteq F \otimes \{0, 1\}$. From the definition of operation $\otimes$, it can be easily shown that $F \otimes \{0, 1\} \supseteq F$. Hence, $F = F \otimes \{0, 1\}$.

2) and 3) can be proved similarly. Q.E.D.

All the closed classes under composition were classified (see Kuntzmann [1]). From the result we can obtain all the closed classes that contain constant functions 0 and/or 1. Fig. 1 shows all the closed classes that contain both 0 and 1. Fig. 2 shows all the closed classes that contain 0, and Fig. 3 shows all the closed classes that contain 1. Figs. 1, 2, and 3 are obtained from Kuntzmann [1]. Therefore, from this and Lemma 1 we can see that the closed classes under degeneration, 0-degeneration, and 1-degeneration are those shown in Figs. 1, 2, and 3, respectively. The notations shown in Figs. 1, 2, and 3 are defined as follows.

A logic function $f(x_1, x_2, \cdots, x_n)$ is said to be *monotone* if for every variable x_i ($1 \leq i \leq n$)

$$f(x_1, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_n) \leq f(x_1, \cdots, x_{i-1}, 1, x_{i+1}, \cdots, x_n).$$

A logic function $f(x_1, \cdots, x_n)$ is said to be *linear* if it can be expressed in the form

$$c_0 \oplus c_1x_1 \oplus \cdots \oplus c_nx_n$$

where $c_i = 0$ or 1 for $i = 0, 1, \cdots, n$. A logic function $f(x_1, \cdots, x_n)$ is said to be an *OR function* if it can be expressed in the form

$$c_0 \vee c_1x_1 \vee \cdots \vee c_nx_n$$

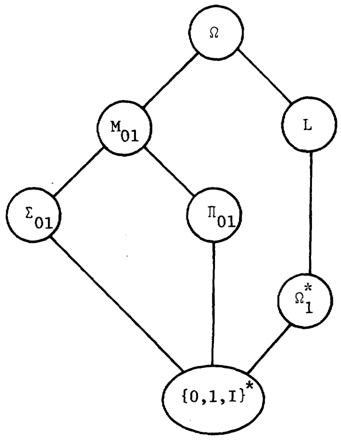

Fig. 1. Closed sets containing constants 0 and 1.

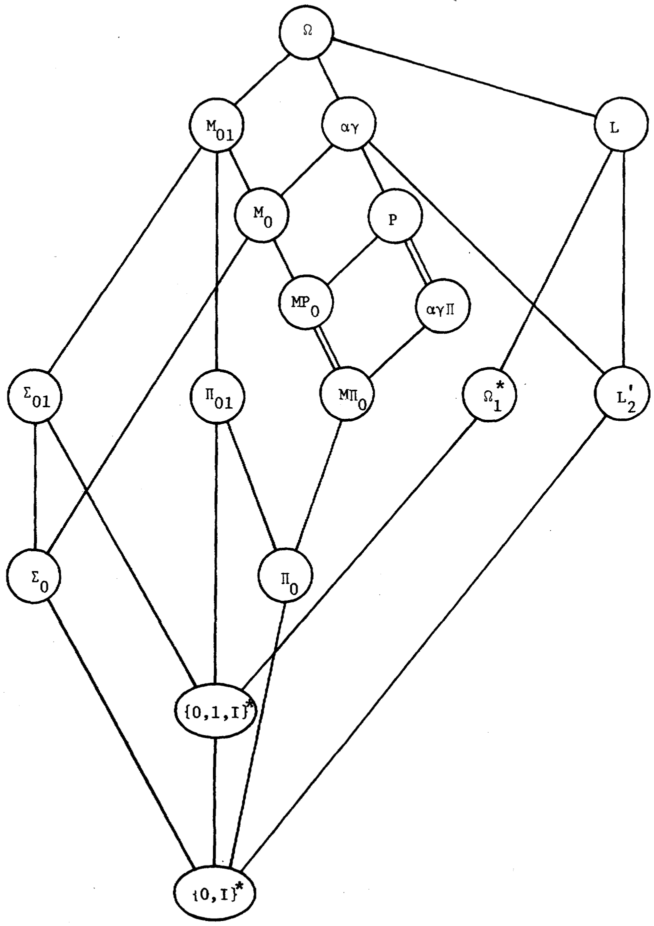

Fig. 2. Closed sets containing constant 0.

where $c_i = 0$ or 1 for $i = 0, 1, \cdots, n$. Similarly, a logic function $f(x_1, \cdots, x_n)$ is said to be an *AND function* if it can be expressed in the form

$$c_0(c_1 \vee x_1)(c_2 \vee x_2)\cdots(c_n \vee x_n)$$

where $c_i = 0$ or 1 for $i = 0, 1, \cdots, n$. A logic function $f(x_1, \cdots, x_n)$ is said to be *self-dual* if the following equation holds:

$$f(\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_n) = \overline{f(x_1, x_2, \cdots, x_n)}.$$

$\alpha = \{f|f(x, x, \cdots, x) = x\}$
$\beta = \{f|f(x, x, \cdots, x) = 1\}$

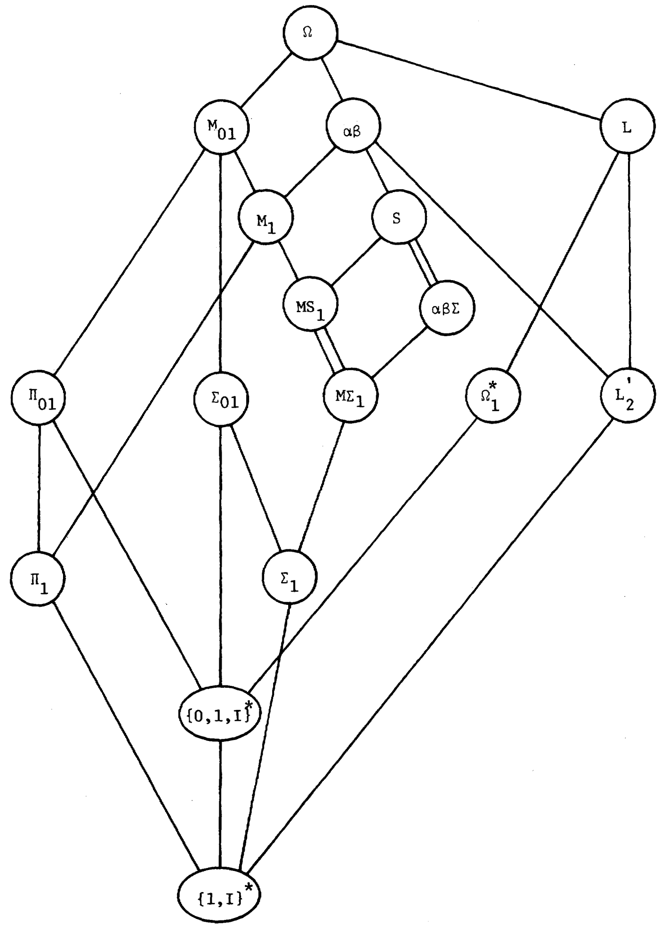

Fig. 3. Closed sets containing contant 1.

$\gamma = \{f|f(x, x, \cdots, x) = 0\}$
$M = \{f|f$ is a monotone function except 0 and 1$\}$
$P = \{f|f = g \cdot h$, g is both monotone and self-dual, and $h \in \Omega\}$
$S = \{f|f = g \vee h$, g is both monotone and self-dual, and $h \in \Omega\}$
$\Sigma = \{x \vee y\}^*$ = class of OR functions
$\Pi = \{x \cdot y\}^*$ = class of AND functions
$L = \{f|f$ is a linear function$\}$
$\alpha\beta = \alpha \cup \beta$
$\alpha\gamma = \alpha \cup \gamma$
$M_{01} = M \cup \{0, 1\}$
$M_0 = M \cup \{0\}$
$M_1 = M \cup \{1\}$
$\Sigma_{01} = \Sigma \cup \{0, 1\}$
$\Sigma_0 = \Sigma \cup \{0\}$
$\Sigma_1 = \Sigma \cup \{1\}$
$\Pi_{01} = \Pi \cup \{0, 1\}$
$\Pi_0 = \Pi \cup \{0\}$
$\Pi_1 = \Pi \cup \{1\}$
$\text{Maj}\,(x, y, z) = xy \vee yz \vee zx$
$MP_0 = \{\text{Maj}, 0\}^*$
$M\Pi_0 = \{f|f = x \cdot g$, g is a product of sums without negation$\} \cup \{0\}$
$\alpha\gamma\Pi = \{f|f = x \cdot g$, g is a product of sums$\}$
$L_2' = \{x \oplus y \oplus 1\}^*$
$MS_1 = \{\text{Maj}, 1\}^*$

$M\Sigma_1 = \{f | f = x \vee g, g$ is a sum of products without negation$\} \cup \{1\}$
$\alpha\beta\Sigma = \{f | f = x \vee g, g$ is a sum of products$\}$.

In Figs. 1, 2, and 3, each single line "—" shows that the lower set is a subset of the upper one, and each double line "=" shows that there exist infinitely many closed sets between both ends.

From Lemma 1 and the result shown in Fig. 1, the following theorem holds.

Theorem 1: Let $F \subseteq \Omega$. If $F = F^* = F \otimes \{0, 1\}$, then F is Ω, M_{01}, L, Σ_{01}, Π_{01}, Ω_1^*, or $\{0, 1, I\}^*$.

III. Closedness Under Stuck-Type Faults

Let F be a set of logic functions, and let f be a logic function in F. If a logic circuit which realizes the function f is composed of the elements realizing functions in F, then such a realization is called a *closed composition with respect to F*. Henceforth, we assume that any realization of a function f in F is a closed composition with respect to F. As an example, consider a function f in F such that f can be expressed in the form

$$f(x_1, x_2, x_3, x_4) = g_1(g_1(x_1, x_2), g_2(x_2, x_3, x_4))$$

where g_1 and g_2 is in F. Then we obtain a closed composition as shown in Fig. 4, where E_1 and E_2 are the elements realizing functions g_1 and g_2, respectively.

The standard stuck-type fault model will be assumed, i.e., all faults can be modeled by lines which are stuck at logical 0 or stuck at logical 1, while the functions performed by the elements in the circuit remain unchanged. A fault consisting of a single stuck line is called a *single fault*, and a fault involving one or more stuck lines is called a *multiple fault*. A set of tests which detects all single (multiple) faults in a circuit C is called a *single (multiple) fault detection test set* for C. A set of tests which detects and distinguishes all distinguishable single (multiple) faults in C is called a *single (multiple) fault location test set* for C.

A set F of logic functions is said to be *closed under stuck faults* if any multiple stuck fault changes a function in F to a faulty function in F. F is said to be *closed under stuck-at-0 (stuck-at-1) faults* if any multiple stuck-at-0 (stuck-at-1) fault changes a function in F to a faulty function in F. The closed sets under stuck-at-0 (stuck-at-1) faults will be considered when the logic circuits are composed of the elements such as fail-safe elements in which only a stuck-at-0 (stuck-at-1) fault may occur. Let $S(F)$ be the set of all faulty functions that can occur from each function in F when a multiple stuck fault is present. Let $S_0(F)$ ($S_1(F)$), be the set of all faulty functions that can occur from each function in F when a multiple stuck-at-0 (stuck-at-1) fault is present. Then we can say that F is closed under stuck fault if and only if $S(F) \subseteq F$, and that F is closed under stuck-at-0 (stuck-at-1) faults if and only if $S_0(F) \subseteq F$ ($S_1(F) \subseteq F$). It can easily be shown that any set of logic functions closed under stuck-type faults is also closed under degeneration and vice versa in the following theorem.

Theorem 2: For any subset F of Ω such that $F = F^*$:

1) $F \supseteq S(F)$ if and only if $F = F \otimes \{0, 1\}$
2) $F \supseteq S_0(F)$ if and only if $F = F \otimes \{0\}$
3) $F \supseteq S_1(F)$ if and only if $F = F \otimes \{1\}$.

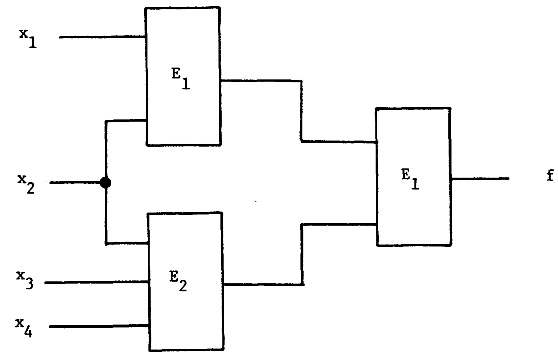

Fig. 4. Circuit realizing function f.

Proof: 1) Suppose that $F \supseteq S(F)$. Obviously, $0, 1 \in S(F)$. Therefore, we have $F \supseteq S(F) \supseteq \{0, 1\}$. From this and Lemma 1, we have $F = F \otimes \{0, 1\}$.

Conversely, suppose that $F = F \otimes \{0, 1\}$, which implies $0, 1 \in F$ from Lemma 1. Let C be a circuit realizing a function f in F. Let σ be a multiple stuck fault such that lines $l_1, l_2, \cdots, l_k$ are stuck at $\sigma_1, \sigma_2, \cdots, \sigma_k$, respectively. Let C_σ be the faulty circuit due to the fault σ, and let f_σ be the faulty function realized by the circuit C_σ. Clearly, the circuit C_σ is equivalent to the circuit in which the lines $l_1, l_2, \cdots, l_k$ are cut and connected with constant elements $\sigma_1, \sigma_2, \cdots, \sigma_k$, respectively. By hypothesis, F is closed under composition, and moreover F contains constants 0 and 1. Therefore, the faulty function f_σ is also in F which implies $S(F) \subseteq F$.

2) and 3) can be proved similarly. Q.E.D.

From Theorems 1 and 2 we have the following corollary.

Corollary 1: Let $F \subseteq \Omega$. If $F = F^*$ and $F \supseteq S(F)$, then F is Ω, M_{01}, L, Σ_{01}, Π_{01}, Ω_1^*, or $\{0, 1, I\}^*$.

Corollary 1 shows that all the sets of logic functions that are closed under both composition and stuck faults are Ω, M_{01}, L, Σ_{01}, Π_{01}, Ω_1^*, and $\{0, 1, I\}^*$. Similarly, from Theorem 2, Lemma 1, and the results shown in Figs. 2 and 3, we can see that all the closed sets under stuck-at-0 faults and stuck-at-1 faults are those shown in Figs. 2 and 3, respectively.

So far, we have considered only the sets closed under composition. However, there still exist many sets of logic functions not closed under composition but closed under stuck faults. Indeed, it can be shown that there exist infinitely many sets of logic functions F such that $F \neq F^*$ and $F \supseteq S(F)$. The classes of fan-out-free functions and unate functions are the examples of such sets of logic functions.

A single-output circuit C is *fan-out-free* if every line in C is connected to an input of at most one gate where the gates are assumed to be AND, OR, NAND, NOR, and NOT gates. A logic function f is said to be a *fan-out-free function* if f can be realized by a fan-out-free circuit. The class of fan-out-free functions is not closed under the composition. As an example, consider three functions f, g_1, and g_2 such that

$$f(x_1, x_2, x_3) = x_1x_2 \vee x_2x_3 \vee x_3x_1$$
$$g_1(x_1, x_2, x_3) = x_1 \vee x_2 \vee x_3$$
$$g_2(x_1, x_2) = x_1x_2.$$

The function f can be expressed using g_1 and g_2 in the form

$$f(x_1, x_2, x_3) = g_1(g_2(x_1, x_2), g_2(x_2, x_3), g_2(x_3, x_1)).$$

Although g_1 and g_2 are fan-out-free functions, f is not fan-out-free. Hence, the class of fan-out-free functions is not closed under composition.

Now we define a restricted composition as follows. Let F be a set of logic functions. If a logic circuit C which realizes a function f in F is composed of the elements realizing functions in F, and if every line in C is connected to any input of at most one element in C, then such a realization is called a *closed tree composition* or simply a *tree composition.*

The class of fan-out-free functions is not closed under composition. However, if we restrict our realization of a given fan-out-free function to a tree composition, then we can easily show that the class of fan-out-free functions is closed under composition, and that it is also closed under stuck faults.

Theorem 3: The class of fan-out-free functions is closed under both composition and stuck faults provided that the circuits under consideration are realized by tree compositions.

It is well known that the class of unate functions is not closed under composition [1]. However, it can be shown that it is closed under some restricted composition. One example of such composition is a *unate gate network* introduced by Reddy [8]. Now we can extend the unate gate network to the closed composition as follows. If a circuit C which realizes a unate function is composed of the elements realizing unate functions and the number of inversions in any path connecting two points in C are the same, then such a realization is called a *unate composition.*

As an example, consider two unate functions g_1 and g_2 such that

$$g_1(x_1, x_2) = x_1 \vee \bar{x}_2 \quad \text{and} \quad g_2(x_1, x_2) = x_1\bar{x}_2.$$

Fig. 5 shows a unate composition using the elements realizing g_1 and g_2, where "+" denotes a positive unate variable and "−" denotes a negative unate variable. The function f realized in Fig. 5 can be expressed as follows:

$$\begin{aligned} f(x_1, x_2, x_3) &= g_1(g_1(x_1, x_2), \overline{g_2(x_2, x_3)}) \\ &= (x_1 \vee \bar{x}_2) \vee \overline{(x_2\bar{x}_3)} \\ &= x_1 \vee \bar{x}_2 \vee x_3. \end{aligned}$$

It can be easily shown that the following theorem holds.

Theorem 4: The class of unate functions is closed under both composition and stuck faults provided that the circuit under consideration is realized by a unate composition.

IV. Test Complexity of Logic Circuits

In this section we will apply the property of the closedness under stuck faults to the fault detection and location problem. For nontrivial sets of logic functions which are closed under stuck faults, i.e., M_{01}, L, Σ_{01}, and Π_{01}, we will consider the test complexity, i.e., the minimum number of fault detection tests and fault location tests, and present the universal test sets derived from the functional description of the circuits.

Given a circuit C, the minimum number of tests required to detect all single faults in C and to detect all multiple faults in C will be denoted by $\delta_s(C)$ and $\delta_m(C)$, respectively. The minimum number of tests required to locate (to within an in-

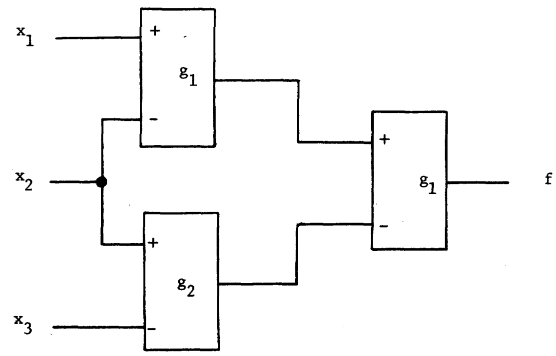

Fig. 5. Unate composition.

distinguishability class) all single faults in C and to locate all multiple faults in C will be denoted by $\lambda_s(C)$ and $\lambda_m(C)$, respectively.

To see the test complexity of the sets of logic functions closed under stuck faults, we first consider the class of monotone functions and unate functions. An input vector X is said to *cover* the input vector Y if X has 1's everywhere Y has 1's. A *minimal true vertex* of a logic function is the input vector that does not cover any other true vertex except itself, and a *maximal false vertex* of a function is the input vector that is not covered by any other false vertex except itself.

Betancourt [5] has shown that the set of all maximal false vertices and minimal true vertices of a unate function f are sufficient to detect any stuck-at-0 or stuck-at-1 fault in any AND/OR realization of f. The result of Betancourt [5] has been extended to multiple faults and unate gate networks by Reddy [8]. The number of maximal false vertices plus minimal true vertices of an n-input unate function is at most

$$\binom{n+1}{\left\lfloor \frac{n+1}{2} \right\rfloor}$$

where $\binom{x}{y}$ denotes the number of combinations choosing y things out of x things and $\lfloor z \rfloor$ denotes the integer part of z. (See Akers [7].) These results can be extended to the unate composition introduced in Section III, as follows.

Theorem 5: If C is any realization of an n-input unate function under unate composition, then

$$\delta_m(C) \le \binom{n+1}{\left\lfloor \frac{n+1}{2} \right\rfloor}$$

This theorem can be proved similarly to the proofs in the literature [7], [8], using "closedness under stuck faults" in Theorem 4, i.e., the property that if one or more stuck-at-0 or stuck-at-1 faults are introduced into a unate circuit under a unate composition, the resulting circuit is still unate.

Corollary 2: If C is any realization of an n-input monotone function under closed composition with respect to M_{01}, then

$$\delta_m(C) \le \binom{n+1}{\left\lfloor \frac{n+1}{2} \right\rfloor}.$$

Next, we will consider the test complexity for the class of linear functions. A linear function $f(x_1, x_2, \cdots, x_n)$ can be expressed in the form

$$c_0 \oplus c_1 x_1 \oplus \cdots \oplus c_n x_n$$

where $c_i = 0$ or 1 for $i = 0, 1, \cdots, n$. If $c_0 = 0$ (1), then the function is called an odd (even) linear function. There are 2^{n+1} linear functions of n variables, of which only two are nondegenerate, i.e.,

$$x_1 \oplus x_2 \oplus \cdots \oplus x_n \quad \text{and} \quad 1 \oplus x_1 \oplus x_2 \oplus \cdots \oplus x_n.$$

The two nondegenerate functions are complements of each other. Hence, it is sufficient to consider only the odd linear functions in order to clarify the test complexity of linear functions. Henceforth, we restrict outselves to odd linear functions which will be denoted by l_n.

Many works on the test complexity of linear functions have been reported [2]–[4]. For any EXCLUSIVE-OR tree realization C of l_n (see Fig. 6), it has been shown that $\delta_s(C) = 4$ by Hayes [2] and $\delta_m(C) \le \left\lfloor \frac{3}{2^n} \right\rfloor + 1$ by Seth and Kodandapani [4], under the fault assumption that any stuck fault within EXCLUSIVE-OR gates is considered. Under the same fault assumption, Hayes [2] has shown that for any EXCLUSIVE-OR cascade realization C of l_n (see Fig. 7), $\delta_m(C) \le n + 2$. Breuer [3] has shown that for any EXCLUSIVE-OR tree realization C of l_n, $\delta_s(C) = 3$ under the fault assumption that permits only stuck faults on the input and output lines of EXCLUSIVE-OR gates.

The linear function is usually realized in either cascade or tree structure of EXCLUSIVE-OR gates. However, these realizations are special cases of closed compositions. There exist some other realizations closed under composition that are neither cascade nor tree. We will consider the test complexity for closed compositions. Then we have the following theorem.

Theorem 6: If C is any realization of l_n under closed composition with respect to L, then $n + 1$ tests $t_0, t_1, \cdots, t_n$ are sufficient to locate or distinguish all distinguishable multiple stuck faults in C, where

$$\begin{aligned} t_0 &= (0, 0, \cdots, 0, 0) \\ t_1 &= (1, 0, \cdots, 0) \\ t_2 &= (0, 1, 0, \cdots, 0) \\ &\vdots \\ t_n &= (0, \cdots, 0, 1). \end{aligned}$$

Proof: Any linear function of n or fewer variables can be expressed in the form

$$l_A(x_1, x_2, \cdots, x_n) = a_0 \oplus a_1 x_1 \oplus a_2 x_2 \oplus \cdots \oplus a_n x_n$$

where $a_i = 0$ or 1 for $i = 0, 1, \cdots, n$ and $A = (a_0, a_1, \cdots, a_n)$. Since the class of linear functions L is closed under composi-

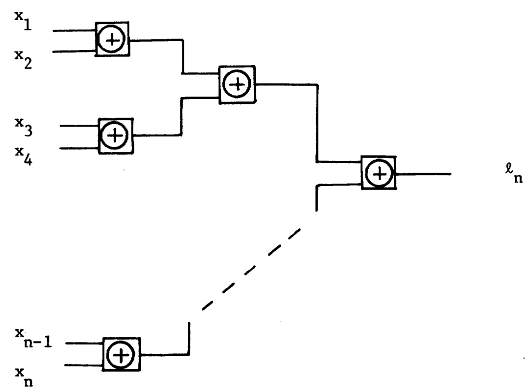

Fig. 6. EXCLUSIVE-OR tree realization.

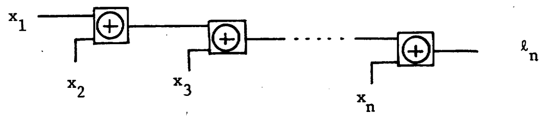

Fig. 7. EXCLUSIVE-OR cascade realization.

tion, any faulty function of linear circuits under closed composition is also a linear function of k ($k \le n$) variables, and thus it can be expressed in the above form. When $A = (0, 1, \cdots, 1)$, the linear function l_A represents the fault-free linear function l_n. Let $l_B = b_0 \oplus b_1 x_1 \oplus \cdots \oplus b_n x_n$ be a linear function realized by a circuit under test where $b_i = 0$ or 1 for $i = 0, 1, \cdots, n$. Applying the vectors $t_0, t_1, \cdots, t_n$ to the equation

$$l_A(x_1, x_2, \cdots, x_n) = l_B(x_1, x_2, \cdots, x_n)$$

we obtain

$$\begin{aligned} a_0 &= b_0 \\ a_0 \oplus a_1 &= b_0 \oplus b_1 \\ &\vdots \\ a_0 \oplus a_n &= b_0 \oplus b_n \end{aligned}$$

which implies

$$a_i = b_i \qquad \text{for } i = 0, 1, \cdots, n.$$

Therefore, we can uniquely determine the values of a_i's, and thus distinguish all 2^{n+1} linear functions l_A's, applying $n + 1$ tests $t_0, t_1, \cdots, t_n$. Q.E.D.

Since the test set $T = \{t_0, t_1, \cdots, t_n\}$ is independent of the structural description of a given circuit, it is a universal test set under closed composition.

Theorem 7: If C is any realization of an n-input linear function under closed composition with respect to L, then

$$\delta_m(C) = \lambda_m(C) = n + 1.$$

Proof: By Theorem 6, $\delta_m(C) \le \lambda_m(C) \le n + 1$. Therefore, it suffices to prove that $\delta_m(C) \ge n + 1$.

Let C be any arbitrary realization of an n-input linear function l_n under closed composition. If it can be shown that $n + 1$ tests are necessary to detect all multiple stuck faults on the primary inputs of C, then we can say $\delta_m(C) \ge n + 1$. This will be proved in the following.

When we consider all the multiple stuck faults on the primary inputs $x_1, x_2, \cdots, x_n$ of C, there exist exactly $2^{n+1} - 2$ faulty functions l_A's such that

$$\left.\begin{array}{l} A = (0, \cdots, 0, 0) \\ (0, \cdots, 0, 1) \\ (0, \cdots, 1, 0) \\ (0, \cdots, 1, 1) \\ \vdots \\ (1, \cdots, 1, 1) \end{array}\right\} 2^{n+1}$$

except $A = (0, 1, \cdots, 1)$ and $(1, 1, \cdots, 1)$. Therefore, l_A is a faulty function if and only if $a_i = 0$ for some i $(1 \le i \le n)$.

Applying all the 2^n input combinations of $(x_1, x_2, \cdots, x_n)$ to the equation $l_A(x_1, x_2, \cdots, x_n) = x_1 \oplus x_2 \oplus \cdots \oplus x_n$, we have 2^n equations as follows:

$$\begin{aligned} a_0 &= 0 \\ a_0 \oplus a_1 &= 1 \\ &\vdots \\ a_0 \oplus a_n &= 1 \\ a_0 \oplus a_1 \oplus a_2 &= 0 \\ a_0 \oplus a_1 \oplus a_3 &= 0 \\ &\vdots \\ a_0 \oplus a_1 \oplus \cdots \oplus a_n &= \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

The maximum number of linearly independent equations among the above 2^n equations is $n + 1$. Therefore, $n + 1$ equations are necessary to determine the values $a_1, a_2, \cdots, a_n$ uniquely. In other words, $n + 1$ tests are necessary to detect all multiple stuck faults on the primary inputs of C. Hence, we obtain $\delta_m(C) \ge n + 1$. Q.E.D.

Next, we will show the test complexity for the class of OR functions and AND functions. Any OR function of n or fewer variables can be expressed in the form

$$f_A(x_1, x_2, \cdots, x_n) = a_0 \vee a_1x_1 \vee a_2x_2 \vee \cdots \vee a_nx_n$$

where $a_i = 0$ or 1 for $i = 0, 1, \cdots, n$ and $A = (a_0, a_1, \cdots, a_n)$. Since the class of OR functions Σ is closed under composition, any faulty function is also an OR function of k $(k \le n)$ variables, and thus it can be expressed in the above form. When $A = (0, 1, \cdots, 1)$, the OR function f_A represents the fault-free OR function. There exist exactly $2^n + 1$ OR functions f_A's. In the same way as the case of linear functions, we can distinguish all $2^n + 1$ OR functions applying $n + 1$ tests $t_0, t_1, \cdots, t_n$ defined in Theorem 6. Hence, we have the following theorem.

Theorem 8: If C is any realization of an n-input OR function under closed composition with respect to Σ, then $n + 1$ tests $t_0, t_1, \cdots, t_n$ are sufficient to locate or distinguish all distinguishable multiple stuck faults in C.

For the class of AND functions we have the dual theorem of Theorem 8 as follows.

Theorem 9: If C is any realization of an n-input AND function under closed composition with respect to Π, then $n + 1$ tests $\bar{t}_0, \bar{t}_1, \cdots, \bar{t}_n$ are sufficient to locate or distinguish all distinguishable multiple stuck faults in C, where

$$\begin{aligned} \bar{t}_0 &= (1, 1, \cdots, 1) \\ \bar{t}_1 &= (0, 1, \cdots, 1) \\ \bar{t}_2 &= (1, 0, 1, \cdots, 1) \\ &\vdots \\ \bar{t}_n &= (1, 1, \cdots, 1, 0). \end{aligned}$$

Theorem 10: If C is any realization of an n-input OR (AND) function under closed composition with respect to $\Sigma(\Pi)$, then

$$\delta_m(C) = \lambda_m(C) = n + 1.$$

The proof of this theorem is similar to Theorem 7.

V. Conclusion

In this paper the concept of closedness under stuck-type faults has been introduced and all the sets of logic functions closed under stuck-type faults have been presented. Applying the property of closedness to the fault detection and location problem, we have clarified the test complexity for the closed classes and presented the universal test sets for them. Nontrivial closed classes have a small sized universal test set due to the closedness under stuck faults. So, the closedness under stuck faults has the ease of detecting and locating faults in the circuits.

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